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The weight of a countably compact group whose cardinality has countable cofinality ☆

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Abstract

We show the existence of a p-compact group whose size has countable cofinality in a forcing model. As a corollary, we show that consistently there exists a countably compact group whose size has countable cofinality and its weight is larger than the size.

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1. Introduction

It is well known that there is a relation between the size of an infinite compact group G and its weight, $w(G)$, given by the equality $|G| = |G|^{w(G)}$ (see [2] for more details). In particular, the cofinality of $|G|$ is uncountable. It is also true for any compact Hausdorff space X that $w(X) \leqslant |X|$.

In [4], van Douwen worked around the statements above, replacing compactness by pseudocompactness, obtaining the following facts:

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(i) (GCH) the cofinality of $|X|$ is uncountable for a homogeneous pseudocompact Tychonoff space X;
(ii) (GCH) if X is a pseudocompact Tychonoff space whose size has countable cofinality then $w(X) = |X|$;
(iii) if $\kappa^\omega = \kappa < \lambda < 2^\kappa$ and λ has countable cofinality then there exists a pseudocompact group G and a countably compact space X whose size is λ and the weight is 2^κ.

The examples in (iii) obtained by van Douwen are not countably compact groups. He showed interest in such examples by asking the following question [4]:

Question 1.5. *If X is an infinite group* (*or a homogeneous space*) *which is countably compact, is $|X|^\omega = |X|$? Is at least* $\mathrm{cf}(|X|) \neq \omega$?

We answered van Douwen's question in the negative in [11]. Indeed, we obtained a forcing model in which there exists a countably compact group of size $\aleph_\omega$. However, the weight of the group is $\mathfrak{c} = \omega_1$. A natural follow up is to construct a countably compact group satisfying (iii).

In this note, we shall give a new answer to van Douwen's question with a stronger compact-like property. As a corollary, we obtain a countably compact topological group whose size has countable cofinality and the weight is larger than the size. We also show that, under an assumption weaker than SCH, if there is an answer to van Douwen's question of size larger than $2^\mathfrak{c}$ then there is one whose weight is larger than the size.

2. Preliminaries

An ultrafilter p in ω^* is selective if for every partition $\{A_n\colon n \in \omega\}$ with $A_n \notin p$ for every $n \in \omega$, there exists $B \in p$ such that $B \cap A_n$ is a singleton. A free ultrafilter p is selective if and only if for every partition $P_0 \cup P_1 \subseteq [\omega]^2$, there exists $A \in p$ and $j < 2$ such that $[A]^2 \subseteq P_j$.

Given two finite sets A and B, we denote by Δ the symmetric difference $A \Delta B = A \setminus B \cup B \setminus A$. The space $([\kappa]^{<\omega}, \Delta)$ is a vector space over 2 and we will often write 'l.i.' instead of 'linearly independent'.

Definition 2.1 [1]. Let X be a topological space and $p \in \omega^*$. A point $x \in X$ is a p-limit point of the sequence $\{x_n\colon n \in \omega\} \subseteq X$ if for every neighborhood U of x, the set $\{n \in \omega\colon x_n \in U\} \in p$.

A sequence $\{x_n\colon n \in \omega\}$ has x as an accumulation point if and only if there exists $p \in \omega^*$ such that x is the p-limit point of the sequence $\{x_n\colon n \in \omega\}$. Given a sequence $\{\{x_n^\alpha\colon \alpha \in I\}\colon n \in \omega\} \subseteq \prod_{\alpha \in I} X_\alpha$, $\{y^\alpha\colon \alpha \in I\}$ is a p-limit of $\{\{x_n^\alpha\colon \alpha \in I\}\colon n \in \omega\}$ if and only if y^α is a p-limit of $\{x_n^\alpha\colon n \in \omega\}$ in X_α for each $\alpha \in I$.

Definition 2.2 [1]. Let $p \in \omega^*$. We say that a topological space X is p-compact if every sequence in X has a p-limit in X.

Every p-compact space is countably compact. Fixed an ultrafilter p, p-compactness is a productive property. In particular, the product of p-compact spaces (for a fixed p) is countably compact.

Countable compactness is not preserved by products in ZFC and under special axioms, not even in the class of topological groups: [3] (Martin's Axiom), [8] (Martin's Axiom for countable partial orders) and [6] (existence of a selective ultrafilter).

Ginsburg and Saks showed [7] that $X^{2^{\mathfrak{c}}}$ is countably compact if and only if X^κ is countably compact for every cardinal κ if and only if X is p-compact for some $p \in \omega^*$.

Saks [10] showed that if p and q are incomparable selective ultrafilters then there exists a p-compact space and a q-compact space whose product is not countably compact. Garcia-Ferreira [5] showed that it is consistent that for any $p, q \in \omega^*$, the product of a p-compact space and a q-compact space is countably compact.

Tomita and Watson [12] showed that if p and q are incomparable selective ultrafilters then there exists a p-compact group and a q-compact group whose product is not countably compact.

Lemma 2.3. *Let p be a selective ultrafilter and E be a countable subset of ordinals. If*

(1) *F is a non-empty finite subset of E;*
(2) *$\{f_m\colon m \in \omega\}$ is a countable family of functions in $([E]^{<\omega})^\omega$ such that for every $k \in \omega$ and $T \subseteq E$ finite, the set $\{n \in \omega\colon \{f_m(n)\colon m \leqslant k\} \cup \{\{\xi\}\colon \xi \in T\}\ l.i.\}$ is an element of p;*
then there exists a homomorphism $\phi : [E]^{<\omega} \to 2$ such that:
(a) *$\phi(F) \neq 0$ and*
(b) *$\{n \in \omega\colon \phi(f_m(n)) = 0\} \in p$ for each $m \in \omega$.*

The proof of Lemma 2.3 is based on a proof in [6]. We include the proof here for the sake of completeness.

Proof. Without loss of generality, assume that $\omega \subseteq E$.

Set $F_0 = F$ and define by induction $\{F_i\colon 1 \leqslant i < \omega\}$ such that

(i) $\bigcup_{i\in\omega} F_i = E$, and
(ii) $F_{i+1} \supseteq F_i \cup \bigcup\{f_m(n)\colon n \leqslant i,\ m \in \omega \cap F_i\}$.

Let $A_i = \{n \in \omega\colon \{f_m(n)\colon m \in \omega \cap F_i\} \cup \{\{\xi\}\colon \xi \in F_i\}$ is $l.i.\}$. By hypothesis, $A_i \in p$ for each $i \in \omega$. Using the selectivity of p, there exists an increasing sequence $A = \{a_i\colon i \in \omega\}$ such that $a_i \in A_i$ and $i < a_i$ for each $i \in \omega$.

Define the following partition $\{P_0, P_1\}$ on $[\omega]^2$, with $m < n$: $\{m, n\} \in P_0$ if and only if there exists $i < j$ such that $\{m, n\} = \{a_i, a_j\}$ and $a_i < j$. By the selectivity of p there exists $B \in p$ such that $B \subseteq A$ and $[B]^2 \subseteq P_0$ or $[B]^2 \subseteq P_1$. Let $\{i_t\colon t \in \omega\}$ be the increasing sequence such that $B = \{a_{i_t}\colon t \in \omega\}$. We claim that $[B]^2$ is a subset of P_0. Indeed, if $[B]^2 \subseteq P_1$ then $a_{i_0} \geqslant i_t$ for every $t \in \omega$, which is a contradiction. Therefore, $a_{i_t} < i_s$ for each $t < s \in \omega$.

Define $E_t = F_{i_t}$ and $b_t = a_{i_t}$ for each $t \in \omega$. Then,

(iii) $\{b_t\colon t \in \omega\} = B \in p$;
(iv) $\{f_m(b_t)\colon m \in \omega \cap E_t\} \cup \{\{\xi\}\colon \xi \in E_t\}$ is linearly independent for each $t \in \omega$; and
(v) $E_{t+1} \supseteq E_t \cup \{f_m(b_t)\colon m \in \omega \cap E_t\}$ for each $t \in \omega$.

Fact (iii) is clear. Fact (iv) follows from $b_t = a_{i_t} \in A_{i_t}$ and $E_t = F_{i_t}$. Fact (v) follows from $b_t = a_{i_t} \leqslant i_{t+1} - 1$ and $E_t = F_{i_t} \supseteq \bigcup\{f_m(n)\colon n \leqslant i_{t+1} - 1,\ m \in \omega \cap F_{i_{t+1}-1}\}$.

We are now ready to construct the homomorphism $\phi : [E]^{<\omega}$ by induction. Since $F \subseteq E_0$, there exists a homomorphism $\phi|_{[E_0]^{<\omega}}$ such that $\phi(F) \neq 0$. Any extension of $\phi|_{[E_0]^{<\omega}}$ will satisfy condition (a). Assume that ϕ is defined on $[E_i]^{<\omega}$ so that

(vi) $\phi(f_m(b_t)) = 0$ for each $m \in \omega \cap E_t$ for each $t < i$.

By condition (iv), the homomorphism $\phi|_{[E_i]^{<\omega}}$ can be easily extended to $[E_{i+1}]^{<\omega}$ so that condition (vi) for $t = i$ is satisfied.

Let $\phi = \bigcup_{i \in \omega} \phi|_{[E_i]^{<\omega}}$. Then clearly ϕ is defined on $[E]^{<\omega}$. We claim that condition (b) holds. Indeed, let $m \in \omega$. Then, there exists $s \in \omega$ such that $m \in E_t$ for each $t \geqslant s$. Thus, by (iii) and (vi), the set $\{n \in \omega\colon \phi(f_m(n)) = 0\}$ contains a cofinite subset of $B \in p$. Therefore, $\{n \in \omega\colon \phi(f_m(n)) = 0\} \in p$ and (b) holds. $\square$

3. The forcing model

Assume CH and let κ be an uncountable cardinal.

Throughout this construction, 2 is the additive group $\mathbb{Z}_2$. Given an ordinal α, the group 2^α is the algebraic product of α copies of the group 2. If $\{z_\mu\colon \mu \in E\}$ is a family contained in 2^α and F is a finite subset of E then $\sum_{\mu \in F} z_\mu$ will be the sum with respect to the binary operation on 2^α. We will often write 'z_F' instead of '$\sum_{\mu \in F} z_\mu$'. Note that if F_0 and F_1 are finite subsets of E then $z_{F_0} + z_{F_1} = z_{F_0 \Delta F_1}$.

Definition 3.1. Let p be an ultrafilter and $\mathcal{F}$ be an infinite subset of $([\kappa]^{<\omega})^\omega$.

We say that $r = (\alpha_r, D_r, \{x^r_\eta\colon \eta \in E_r\})$ is an element of $\mathbb{S}_{p,\mathcal{F}}$ if $\alpha_r \in \omega_1$, $D_r \in [\mathcal{F}]^\omega$, $E_r \in [\kappa]^\omega$ with $\bigcup_{f \in D_r, n \in \omega} f(n) \subseteq E_r$ and $x^r_\eta \in 2^{\alpha_r}$ for each $\eta \in E_r$. Given $r, s \in \mathbb{S}_{p,\mathcal{F}}$, we say that $r \leqslant s$ if $\alpha_r \geqslant \alpha_s$, $D_r \supseteq D_s$, $E_r \supseteq E_s$, $\forall \eta \in E_s (x^r_\eta|_{\alpha_s} = x^s_\eta)$ and the sequence $\{x^r_{f(n)}|_{[\alpha_s, \alpha_r)}\colon n \in \omega\}$ has $0|_{[\alpha_s,\alpha_r)} \in 2^{[\alpha_s,\alpha_r)}$ as p-limit for every $f \in D_s$.

Lemma 3.2. *The set $\mathbb{S}_{p,\mathcal{F}}$ endowed with the partial ordering above is countably closed and ω_2-cc.*

Proof. Given a decreasing sequence $\{r_n\colon n \in \omega\}$ with $r_n = (\alpha_n, D_n, \{x^n_\eta\colon \eta \in E_n\})$, let $r_\omega = (\alpha_\omega, D_\omega, \{x^\omega_\eta\colon \eta \in E_\omega\})$, where $\alpha_\omega = \bigcup_{n \in \omega} \alpha_n$, $D_\omega = \bigcup_{n \in \omega} D_n$, $E_\omega = \bigcup_{n \in \omega} E_n$ and $x^\omega_\eta = \bigcup_{n \in \omega \wedge \eta \in E_n} x^n_\eta$.

Clearly $r_\omega \in \mathbb{S}_{p,\mathcal{F}}$ and $r_\omega \leqslant r_n$ for each $n \in \omega$. Hence, $\mathbb{S}_{p,\mathcal{F}}$ is countably closed.

Let $\{r_\mu\colon \mu < \omega_2\}$ be a subset of $\mathbb{S}_{p,\mathcal{F}}$, where $r_\mu = (\alpha_\mu, D_\mu, \{x^\mu_\eta\colon \eta \in E_\mu\})$ for each $\mu < \omega_2$. Using the Δ-system lemma and CH, there exists $E \in [\kappa]^{\leqslant\omega}$ and $I \in [\omega_2]^{\omega_2}$ such that $E_\mu \cap E_\beta = E$ for any pair $\{\mu, \beta\} \in [I]^2$. We can also assume that there exists $\alpha \in \omega_1$

such that $\alpha_\mu = \alpha$ for every $\mu \in I$. Furthermore, there are only $\mathfrak{c} = \omega_1$ functions from E to 2^α, thus, we can assume that for every pair $\{\mu, \beta\} \in [I]^2$ $x_\eta^\beta = x_\eta^\mu$ for every $\eta \in E$.

Fix $\mu, \beta \in I$ distinct and let $r = (\alpha, D_\mu \cup D_\beta, \{x_\eta^\mu\colon \eta \in E_\mu\} \cup \{x_\eta^\beta\colon \eta \in E_\beta \setminus E\})$. Then $r \leqslant r_\beta$ and $r \leqslant r_\mu$. Therefore, $\mathbb{S}_{p,\mathcal{F}}$ has the ω_2-cc property. $\square$

We define now some subsets of $\mathbb{S}_{p,\mathcal{F}}$ that will be dense for appropriate ultrafilters p and families $\mathcal{F}$.

Definition 3.3. Let $F \in [\kappa]^{<\omega} \setminus \{\emptyset\}$, $f \in \mathcal{F}$ and $\alpha \in \omega_1$. Define $\mathcal{D}_{F,f,\alpha} = \{r \in \mathbb{S}_{p,\mathcal{F}}\colon F \subseteq E_r,\ f \in D_r \wedge (\exists \gamma \in [\alpha, \alpha_r))x_F^r(\gamma) \neq 0\}$.

Definition 3.4. Given a finite set $F \in [\kappa]^{<\omega}$, define $\vec{F} : \omega \to [\kappa]^{<\omega}$ be the constant function $\vec{F}(n) = F$ for each $n \in \omega$. If $\alpha < \kappa$ we shall write $\vec{\alpha}$ instead of $\{\vec{\alpha}\}$.

The following is the definition of the ultrapower of the group $([\kappa]^{<\omega}, \Delta)$ with respect to p:

Definition 3.5. Given $p \in \omega^*$ and $f \in ([\kappa]^{<\omega})^\omega$, define $[f]_p = \{g \in ([\kappa]^{<\omega})^\omega\colon \{n \in \omega\colon f(n) = g(n)\} \in p\}$. Let $([\kappa]^{<\omega})^\omega/p = \{[f]_p\colon f \in ([\kappa]^{<\omega})^\omega\}$ be the vector space over 2 under the operation $[f]_p \Delta [g]_p = [f \Delta g]_p$, where $(f \Delta g)(n) = f(n) \Delta g(n)$ for each $n \in \omega$.

Lemma 3.6. *The sets defined in Definition* 3.3 *are dense in* $\mathbb{S}_{p,\mathcal{F}}$ *if* p *is a selective ultrafilter and* $\{[f]_p\colon f \in \mathcal{F}\} \cup \{[\vec{\alpha}]_p\colon \alpha < \kappa\}$ *is linearly independent.*

Proof. Let r be an arbitrary element of $\mathbb{S}_{p,\mathcal{F}}$ and fix $F \in [\kappa]^{<\omega}$, $f \in \mathcal{F}$ and $\alpha \in \omega_1$. Let t be defined as follows: $\alpha_t = \max\{\alpha, \alpha_r\}$, $D_t = D_r \cup \{f\}$, $E_t = E_r \cup F \cup \bigcup\{f(n)\colon n \in \omega\}$. For each $\eta \in E_t \setminus E_r$, define $x_\eta^t = 0 \in 2^{\alpha_t}$ and for each $\eta \in E_r$, define $x_\eta^t = x_\eta^r \cup 0|_{[\alpha_r, \alpha_t)} \in 2^{\alpha_r} \times 2^{\alpha_t \setminus \alpha_r}$. Clearly $t \leqslant r$. We will extend t to s with $s \in \mathcal{D}_{F,f,\alpha}$.

Apply Lemma 2.3 to obtain a homomorphism $\phi : [E_t]^{<\omega} \to 2$ such that

(1) $\phi(F) \neq 0$, and
(2) p-$\lim\{\phi(g(n))\colon n \in \omega\} = 0$ for every $g \in D_t$.

Define $D_s = D_t$, $E_s = E_t$ and $\alpha_s = \alpha_t + 1$. Define $x_\eta^s = x_\eta^t \cup \{(\alpha_t, \phi(\{\eta\}))\}$. It follows from property (1) that $s \in \mathcal{D}_{F,f,\alpha}$ and it follows from property (2) that $s \leqslant t$. $\square$

We recall that a forcing preserves cardinals if every cardinal in the ground model is a cardinal in the extension and a forcing preserves cofinalities if the cofinality of a cardinal is the same ordinal in the ground model and in the extension.

Lemma 3.7. *Assume* CH. *Let* $\kappa > \lambda > \mathfrak{c}$ *be cardinals,* p *be a selective ultrafilter,* $\mathcal{F}$ *be a family of sequences in* $[\kappa]^{<\omega}$ *such that* $\{[f]_p\colon f \in \mathcal{F}\}$ *is linearly independent and* $\mathbb{S}_{p,\mathcal{F}}$ *be the partial order defined in Definition* 3.1. *Then* $\mathbb{S}_{p,\mathcal{F}}$ *preserves cardinals and cofinalities and* $2^{\omega_1} > \lambda$ *in the extension. Furthermore, if* $S \subseteq \kappa$ *and* $\mathcal{F}' \subseteq \mathcal{F}$ *are such that* $|S| = \lambda$

and $\{[f]_p\colon f \in \mathcal{F}'\} \cup \{[\vec{\gamma}]_p\colon \gamma \in S\}$ *is a basis for the vector space* $([S]^{<\omega})^\omega/p$ *then there exists a* p*-compact group of size* λ *in the extension.*

Proof. From Lemma 3.2, the partial ordering $\mathbb{S}_{p,\mathcal{F}}$ is countably closed and ω_2-cc. Thus, it preserves cardinals and cofinalities. Therefore, $\lambda < \kappa$ are cardinals in the extension. Let $\mathcal{G}$ be a generic filter for the partial order $\mathbb{S}_{p,\mathcal{F}}$ which intersects each dense set in Definition 3.3. For each $\xi \in \kappa$, let $x_\xi = \bigcup_{r \in \mathcal{G} \wedge \xi \in E_r} x^r_\xi$.

Fix $f_0 \in \mathcal{F}$.

For every $\xi < \kappa$ and $\alpha < \omega_1$ there exists $r \in \mathcal{G} \cap \mathcal{D}_{\{\xi\}, f_0, \alpha}$ such that $\alpha_r > \alpha$ and $\xi \in E_r$. Therefore, $x_\xi \in 2^{\omega_1}$.

We claim that $\{x_\xi\colon \xi < \kappa\}$ is linearly independent. Indeed, if F is a non-empty finite subset of κ, there exists $r \in \mathcal{G} \cap \mathcal{D}_{F, f_0, 0}$. Therefore, there exists $\gamma < \alpha_r$ such that $x_F(\gamma) = x^r_F(\gamma) \neq 0 \in 2$. Thus, x_F is not the neutral element of 2^{ω_1}.

From the previous claim, 2^{ω_1} contains at least κ different functions, thus, $2^{\omega_1} \geqslant \kappa > \lambda$.

Let H be the group $\bigcup_{\alpha < \omega_1} 2^\alpha \times \{0\}^{\mathfrak{c} \setminus \alpha}$. We claim that p-$\lim\{x_{f(n)}\colon n \in \omega\} \in H$ for each $f \in \mathcal{F}$. Indeed, let $f \in \mathcal{F}$ and let $r \in \mathcal{G} \cap \mathcal{D}_{\emptyset, f, 0}$. Then, for each $\beta > \alpha_r$ there exists $s \in \mathcal{G}$ such that $s \leqslant r$ and $\beta < \alpha_s$. Thus, p-$\lim\{x_{f(n)}(\beta)\colon n \in \omega\} = p$-$\lim\{x^s_{f(n)}(\beta)\colon n \in \omega\} = 0$. Therefore, p-$\lim\{x_{f(n)}\colon n \in \omega\} \in 2^{\alpha_r} \times \{0\}^{\mathfrak{c} \setminus \alpha_r} \subseteq H$.

Let S and $\mathcal{F}'$ be as in the hypothesis of the lemma. Clearly the group G_S generated by $\{x_\xi\colon \xi \in S\} \cup H$ has size λ, since H has size $\mathfrak{c}$ and S has size $\lambda > \mathfrak{c}$. We will show now that G_S is p-compact. Let $\{y_n\colon n \in \omega\}$ be a sequence in G_S. Then there exists a sequence $\{F_n\colon n \in \omega\} \subseteq [S]^{<\omega}$ such that $\{y_n - x_{F_n}\colon n \in \omega\} \in H$. The group H is ω-bounded, thus p-compact. Therefore, the sequence $\{y_n - x_{F_n}\colon n \in \omega\}$ has a p-limit in $H \subseteq G_S$. It suffices then to show that $\{x_{F_n}\colon n \in \omega\}$ has also a p-limit in G_S. Since we are forcing with a countably closed partial ordering, the set $\{F_n\colon n \in \omega\}$ is in the ground model. By the hypothesis on $\mathcal{F}'$, there exists $A \in p$, $D \in [\mathcal{F}']^{<\omega}$ and $T \in [S]^{<\omega}$ such that $F_n = (\Delta_{f \in D} f(n)) \Delta T$ for each $n \in A$. Then, $x_{F_n} = \sum_{f \in D} x_{f(n)} + x_T$ for each $n \in A$. Therefore, p-limit $\{x_{F_n}\colon n \in \omega\} = \sum_{f \in D} p$-$\lim x_{f(n)} + x_T \in H + x_T \subseteq G_S$. Thus, every sequence in G_S has a p-limit. □

Lemma 3.8. *Let* θ *and* λ *be cardinals satisfying* $\theta^\omega = \theta < \lambda < 2^\theta$ *and* p *be an ultrafilter. If there exists a* p*-compact group* G *of size* λ *then there exists a* p*-compact group of size* λ *and weight* $2^\theta > \lambda$.

Proof. Let Z be a dense subset of 2^{2^θ} of size $\mathfrak{c}$. Doing a closing-off argument, it is easy to see that there exists a group $T \supseteq Z$ such that T has size $\theta^\omega = \theta$, is p-compact and $w(T) = 2^{\mathfrak{c}}$. Then, $G \times T$ is a p-compact group whose size is λ and the weight is 2^θ. □

The next theorem shows the result in the abstract.

Theorem 3.9. *It is consistent that there exists an ultrafilter* p *and a* p*-compact group whose size has countable cofinality and the weight is larger than the size of the group.*

Proof. Assume CH. Fix a selective ultrafilter p and a cardinal λ of countable cofinality and a cardinal κ greater than λ. Let $\mathcal{F}$ be a family of sequences in $[\lambda]^{<\omega}$ such that $\{[f]_p\colon f \in$

$\mathcal{F}\} \cup \{[\vec{\gamma}]_p\colon \gamma < \lambda\}$ is a basis for the vector space $([\lambda]^{<\omega})^\omega/p$. Then, by Lemma 3.7, there exists in the extension a p-compact group of size λ. Since $\omega_1^\omega = \omega_1 < \lambda < 2^{\omega_1}$, there exists, by Lemma 3.8, a p-compact group of size λ and weight $2^{\omega_1} > \lambda$. This group is as required, since λ has countable cofinality in the extension. □

In the example above, for each $\lambda < \kappa$ whose cardinality has countable cofinality, we can force a p-compact group topology on the Boolean group of size λ. We modify the example above to show that we can obtain a p-compact group topology for each Boolean group of size λ for each $\lambda < \kappa$ in a single forcing model. It is easy to obtain p-compact groups whose size is $\lambda = \lambda^\omega$, thus, the important feature of the example below is that $2^{\mathfrak{c}}$ "arbitrarily large" will guarantee that there are plenty of cardinals of countable cofinality below $2^{\mathfrak{c}}$.

Theorem 3.10. *It is consistent that $2^{\mathfrak{c}}$ is "arbitrarily large" and for every $\lambda \in [\mathfrak{c}, 2^{\mathfrak{c}}]$ there exists a p-compact group of size λ and weight larger than λ.*

Proof. Assume CH, and let p be a selective ultrafilter and κ be a cardinal with $\kappa^{\omega_1} = \kappa$. As noted before, it is easy to construct the required example for $\lambda = 2^{\mathfrak{c}}$, since $(2^{\mathfrak{c}})^\omega = 2^{\mathfrak{c}}$. Let J be the set of all cardinals λ with $\omega_1 \leqslant \lambda < \kappa$. We will show that $\omega_1 = \mathfrak{c}$ and $\kappa = 2^{\mathfrak{c}}$ in the extension and produce an example for each $\lambda \in J$.

Let $\{S_\lambda\colon \lambda \in J\}$ be a partition of κ with $|S_\lambda| = \lambda$ for each $\lambda \in J$. Let $\{\mathcal{F}_\lambda\colon \lambda \in J\}$ be a family of sequences in $[S_\lambda]^{<\omega}$ such that $\{[f]_p\colon f \in \mathcal{F}_\lambda\} \cup \{[\vec{\beta}]_p\colon \beta \in S_\lambda\}$ is a basis for $(([S_\lambda]^{<\omega})^\omega)/p$. Let $\mathcal{F} = \bigcup_{\lambda\in J} \mathcal{F}_\lambda$. Clearly $\{[f]_p\colon f \in \mathcal{F}\} \cup \{[\vec{\beta}]_p\colon \beta < \kappa\}$ is a linearly independent subset of $(([\kappa]^{<\omega})^\omega)/p$. Applying Lemma 3.7 on each $\mathcal{F}_\lambda$ in place of $\mathcal{F}'$, we show that there exists a p-compact group of size λ for each $\lambda < \kappa$. Applying Lemma 3.8, there exists a p-compact group of size λ and weight greater than λ for each cardinal $\lambda < \kappa$.

To obtain the required conclusion, it suffices to show that $2^{\mathfrak{c}} = \kappa$. CH holds in the ground model and the forcing is countably closed, thus, no new countable subsets of ω are added in the extension. Therefore, CH still holds in the extension and $2^{\omega_1} = 2^{\mathfrak{c}}$. We have already seen in Lemma 3.7 that $2^{\omega_1} \geqslant \kappa$. Thus, we have only to show that $2^{\omega_1} \leqslant \kappa$. The partial order $\mathbb{S}_{p,\mathcal{F}}$ is ω_2-cc and has size κ. Thus, there are $\kappa = \kappa^{\omega_1}$ nice $\mathbb{S}_{p,\mathcal{F}}$-names for subsets of ω_1 (see [9, pp. 208 and 215]). Therefore, $2^{\omega_1} \leqslant \kappa$ in the extension and we are done. □

4. Reviving van Douwen's question

The known examples of countably compact groups whose size has countable cofinality have size at most $2^{\mathfrak{c}}$. Thus, van Douwen's question can be revived as follows:

Question 4.1. *If X is an infinite group (or a homogeneous space) of size greater than $2^{\mathfrak{c}}$ which is countably compact, is $|X|^\omega = |X|$? Is at least $\mathrm{cf}(|X|) \neq \omega$?*

If there is a countably compact group G answering the question above in the negative and there exists $\theta = \theta^\omega < |G| < 2^\theta$ then there is also one whose weight is greater than its size:

Proposition 4.2. *Let λ be a cardinal of countable cofinality such that $2^{\mathfrak{c}} < \lambda$ and $\theta^\omega < \lambda$ for each cardinal $\theta < \lambda$. If there exists a countably compact Abelian group G of size λ then there exists one of size λ and weight greater than λ.*

Proof. Suppose that there exists a countably compact group of size λ. By a result of van Douwen [4] there exists $\kappa < \lambda$ such that $2^\kappa > \lambda$. We can assume that $\theta = \kappa^\omega < \lambda$. Thus, $\theta^\omega = \theta < \lambda < 2^\theta$.

Let H_0 be a dense subgroup of $K = 2^{2^\theta}$ of size θ. Let $H = \bigcup_{E \in [H_0]^\omega} \overline{\langle E \rangle}$. Then H is an ω-bounded group of size θ^ω. $2^{\mathfrak{c}} = \theta$ and weight 2^θ. The product of an ω-bounded space and a countably compact space is countably compact. Therefore, the group $G \times H$ is countably compact and has size λ and weight greater than λ. $\square$

The fact that $\lambda > 2^{\mathfrak{c}}$ is essential in the proof of Proposition 4.2, since a separable Hausdorff group can have size up to $2^{\mathfrak{c}}$. This suggests the following:

Question 4.3. *If $\lambda < 2^{\mathfrak{c}}$ has countable cofinality and there exists a countably compact group of size λ, is there a countably compact group of size λ whose weight is larger than λ*?

Our p-compact groups whose size have countable cofinality have convergent sequences, and the known examples of countably compact groups whose size has countable cofinality and have no non-trivial convergent sequences have weight $\omega_1 = \mathfrak{c}$. The following questions remain open.

Question 4.4. *Is there a countably compact* (*or even a p-compact*) *topological group G without non-trivial sequences such that $w(G) > |G|$ and* $\mathrm{cf}(|G|) = \omega$?

Question 4.5. *If there exists a countably compact group G such that $|G| > 2^{\mathfrak{c}}$,* $\mathrm{cf}(|G|) = \omega$ *and G has no non-trivial convergent sequences then is there a countably compact group H that, in addition, has weight larger than the size*?

We point out that Proposition 4.2 cannot be used to answer Question 4.5, since ω-bounded subgroups contain convergent sequences.

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